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An Implicit-Function Theorem for $C^{0,1}$-Equations and Parametric $C^{1,1}$-Optimization

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The implicit-function theorem deals with the solutions of the equation $F(x, t) = a$ for locally Lipschitz functions F from R^{n+m} into R^n. The existence of a locally well-defined and Lipschitzian solution function $x = G(a, t)$ will be completely characterized in terms of certain multivalued directional derivatives of F which determine the corresponding derivatives of G in a simple way. Our directional derivatives are nothing but L. Thibault's (*Ann. Mat. Pura Appl. (4)* **125**, 1980, 157–192) limit sets which have been introduced to extend Clarke's calculus to functions in abstract spaces. For parametric $C^{1,1}$-optimization problems, we study the critical point map, the associated critical values, and derive first and second order formulas, respectively.

1. INTRODUCTION

In order to study the implicit function $x = x(t)$ of the equation

$$F(x, t) = 0, \tag{1.1}$$

where F is locally Lipschitz, various ideas are to be found in many publications. The introduction in [23] gives an impressive survey about recent contributions to this problem area including generalized equations, variational inequalities, and maps in infinite-dimensional spaces.

By our references which are far from being complete, we hope to reflect at least main developments of this field. Having in mind the C^1-version of the implicit-function theorem the framework of an ideal corresponding theorem for locally Lipschitz F is obvious.

First, define some derivative DF for these functions. Next, verify that, around some zero $(\bar{x}, \bar{t})$ of F, there is a locally well-defined and Lipschitz implicit function $x = x(t)$ if and only if $DF(\bar{x}, \bar{t})$ is non-degenerated in a certain sense. Finally, determine $Dx(\bar{t})$ in terms of $DF(\bar{x}, \bar{t})$. To verify such

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a theorem creates several difficulties although there are various sufficient conditions for $x = x(t)$ to be locally well-defined and Lipschitz, see F. H. Clarke's [3, 4] inverse and implicit function theorem and alternative concepts which use strong B-derivatives [18, 23] and different approximations of multifunctions [1, 2, 7, 12, 20].

In what follows, it turns out that L. Thibault's limit sets $D_h(x; u)$, which are defined in [24] for so-called compactly Lipschitzian functions h in topological vector spaces, play an important role in the present context. We will denote them by $\Delta h(x; u)$, and call these sets directional derivatives. The motivation for doing so comes from the facts that we will need partial "derivatives" and that these sets shall take the place of $Dh(x)\,u$ in the C^1-case.

Instead of Eq. (1.1) we have to consider the equation $F(x, t) = a$ and the related implicit function $x = G(a, t)$. This device is needed since the formula $\Delta F = \Delta_x F + \Delta_t F$ (partial derivatives) does not hold in general.

The proof of the implicit-function theorem forms the content of the next section. Thereafter, we apply M. Kojima's [15] characterization of critical points for optimization problems (as zeros of a related Lipschitz function) and specify the implicit-function theorem to this particular case. Concerning the involved functions we will need a $C^{1,1}$-property w.r. t. primal variables and a certain C^2-dependence on the parameters. Our result consists in a complete characterization of the regular (also called strongly stable) case, in second (more precise 1.1) order formulas for the marginal function, and some insight into the strict complementarity. This way we shall extend and generalize similar results for C^2-problems [6, 9, 10, 15, 21] by a unified approach. For $C^{1,1}$-problems, which are of particular interest in two-level optimization [11, 22] and for continuation methods [8, 19], generalized Jacobians [5, 10, 13] as well as B-derivatives [23] have been used in order to derive sufficient regularity conditions. It looks, however, very hard to establish some marginal-value theory on these statements.

Finally, it should be noted that the present concept, contrary to B-derivatives or contingent cones, makes sense only in the case of finite dimension.

Notations. Given a function f mapping some open subset Ω of R^n into R^m we write $f \in C^{0,1}$ ($f \in C^{1,1}$) to indicate that f is locally Lipschitz (Df exists and is locally Lipschitz). If the domain is of particular interest we will also write $f \in C^{0,1}(\Omega, R^m)$. Set-valued operations as $A + B$, $\langle a, B \rangle$ denote, as usual, the union of the corresponding elementwise results. By $B(x, \varepsilon)$ the closed ball with center x and radius ε is denoted.

The Implicit-Function Theorem

Our main tool is the following definition of a set-valued directional derivative of a function $h \in C^{0,1}(R^p, R^q)$ at some point x in direction u: The set $\Delta h(x; u)$ consists of all limits

$$v = \lim(h(x^k + \lambda_k u) - h(x^k))/\lambda_k, \qquad \text{where} \quad x^k \to x \text{ and } \lambda_k \searrow 0. \tag{2.1}$$

If $q = 1$, these limits are known from Clarke's directional derivatives, and $\Delta h(x, u)$ coincides with $\partial h(x)\, u$ where ∂h denotes Clarke's generalized Jacobian.

For $q > 1$ these sets have been introduced and studied in [24] in order to extend Clarke's calculus to functions in general spaces. Particularily, the following basic properties are shown to hold in [24, 25].

The map $(x, u) \mapsto \Delta h(x; u)$ is closed, locally bounded, and fulfills $\Delta h(x; u_1 + u_2) \subset \Delta h(x; u_1) + \Delta h(x; u_2)$, $\Delta h(x; \lambda u) = \lambda \Delta h(x; u)$ $(\lambda \in R)$, $\operatorname{conv} \Delta h(x; u) = \partial h(x)\, u$.

In [16], we have shown that Rademacher's theorem is not needed in order to establish, on the present sets, some calculus including chain-rules and mean-value theorems. Example 2 in [16] makes obvious that $\Delta h(x; u) \neq \partial h(x)\, u$ may happen for a piecewise linear homeomorphism of R^2 into itself and that $\Delta h(x; u)$ may be nonconvex. Since, in this example, the zero-matrix belongs to $\partial h(0)$, Clarke's supposition of non-singularity of $\partial h(x)$ turns out to be no necessary condition in his inverse function theorem [3].

Now, let $F \in C^{0,1}(R^{n+m}, R^n)$ be given and suppose $F(\bar{x}, \bar{t}) = \bar{a}$. Let $N(\bar{x})$ and $N(\bar{a}, \bar{t})$ be some neighborhoods of $\bar{x} \in R^n$ and $(\bar{a}, \bar{t}) \in R^{n+m}$, respectively. We consider the equation

$$F(x, t) = a; \qquad x \in N(\bar{x}),\ (a, t) \in N(\bar{a}, \bar{t}) \tag{2.2}$$

and call the function F (or Eq. (2.2)) *regular* at $(\bar{x}, \bar{t}, \bar{a})$ if there are such neighborhoods that, to each $(a, t) \in N(\bar{a}, \bar{t})$, corresponds a unique solution $x = G(a, t)$ of (2.2) and if, additionally, G is Lipschitz on $N(\bar{a}, \bar{t})$.

THEOREM 1. *The function F is regular at $(\bar{x}, \bar{t}, \bar{a})$ if and only if*

$$0 \notin \Delta F((\bar{x}, \bar{t}); (u, 0)) \qquad \textit{for each } u \in R^n \setminus \{0\}. \tag{2.3}$$

If (2.3) *holds true then the inclusions*

$$u \in \Delta G((\bar{a}, \bar{t}); (\alpha, \tau)) \tag{2.4}$$

and

$$\alpha \in \Delta F((\bar{x}, \bar{t}); (u, \tau)) \tag{2.5}$$

are equivalent.

Before proving the theorem regard some special cases.

(i) $F \in C^1$. Then, the directional derivatives $\Delta F((x, t); (u, \tau))$ are single-valued and consists of $D_x F(x, t) u + D_t F(x, t) \tau$. Condition (2.3) means regularity of the matrix $D_x F(\bar{x}, \bar{t})$, and the equivalence yields $\Delta G((\bar{a}, \bar{t}); (\alpha, \tau)) = (D_x F(\bar{x}, \bar{t}))^{-1}(\alpha - D_t F(\bar{x}, \bar{t}) \tau)$.

(ii) $F(x, t) \equiv F(x)$. Condition (2.3) is reduced to $0 \notin \Delta F(\bar{x}; u)\ \forall u \in R^n \backslash \{0\}$ and means, by the theorem, that the inverse F^{-1} is locally well-defined and Lipschitz, and $v \in \Delta F(\bar{x}; u)$ iff $u \in \Delta F^{-1}(\bar{a}; v)$. This way we obtain the inverse function theorem of [16], and via $\Delta h(x; u) \subset \partial h(x) u$ the corresponding theorem of Clarke. The inclusion $\Delta h(x; u) \subset \partial h(x) u$ makes a non-trivial statement since, for proving it, Rademacher's theorem has to be used (or more directly [16] Clarke's mean-value theorem).

(iii) Define partial derivatives $\Delta_x F$, $\Delta_t F$ by fixing, in the usual way, the remaining variables, and suppose

$$\Delta F((\bar{x}, \bar{t}); (u, \tau)) = \Delta_x F((\bar{x}, \bar{t}); u) + \Delta_t F((\bar{x}, \bar{t}); \tau). \tag{2.6}$$

Now, (2.3) takes the form $0 \notin \Delta_x F((\bar{x}, \bar{t}); u)\ \forall u \neq 0$ and says that $\Phi = (F(., \bar{t}))^{-1}$ is locally well-defined and Lipschitz (near $\bar{a}$). Moreover, the inverse function completely determines ΔG as

$$\Delta G((\bar{a}, \bar{t}); (\alpha, \tau)) = \Delta \Phi(\bar{a}; \alpha - \Delta_t F((\bar{x}, \bar{t}); \tau)).$$

Obviously, this case is the set-valued version of (i).

Proof of Theorem 1. In the *first part*, we will verify that condition (2.3) implies regularity of F at $(\bar{x}, \bar{t}, \bar{a})$.

Step 1. Condition (2.3) is equivalent to the existence of some positive ε such that

$$\|F(x'', t'') - F(x', t')\| \geqslant \varepsilon(\|x'' - x'\| + \|t'' - t'\|) \tag{2.7}$$

whenever $x'', x' \in B(\bar{x}, \varepsilon)$; $t'', t' \in B(\bar{t}, \varepsilon)$, and $\|t'' - t'\| \leqslant \varepsilon \|x'' - x'\|$.

Indeed, if (2.7) is true then each limit

$$v = \lim (F(x^k + \lambda_k u, t^k) - F(x^k, t^k))/\lambda_k,$$

where $x^k \to \bar{x}$, $t^k \to \bar{t}$, $\lambda_k \searrow 0$, fulfills $\|v\| \geqslant \varepsilon \|u\|$. On the other hand, if (2.7) is false for each $\varepsilon > 0$, we may consider any sequence $\varepsilon \searrow 0$ and related x'',

x', t'', t' satisfying the negation of (2.7). Then, $x'' \neq x'$ is evident. Setting $\lambda(\varepsilon) = \|x'' - x'\|$ and $u(\varepsilon) = (x'' - x')/\lambda(\varepsilon)$ the sequence $u(\varepsilon)$ may be assumed to converge $u(\varepsilon) \to u \neq 0$. Since F is Lipschitz near $(\bar{x}, \bar{t})$, say with rank L, we thus obtain

$$\|F(x' + \lambda(\varepsilon)\, u, t') - F(x', t')\| \leqslant \|F(x'', t'') - F(x', t')\| + L(\|t'' - t'\| + \lambda(\varepsilon)\|u(\varepsilon) - u\|) < \lambda(\varepsilon)(2\varepsilon + L(\varepsilon + \|u(\varepsilon) - u\|))$$

and $0 \in \Delta F((\bar{x}, \bar{t}); (u, 0))$.

Step 2. Suppose (2.7) and assign, to $a \in R^n$ and $t \in B(\bar{t}, \varepsilon)$, any $x \in B(\bar{x}, \varepsilon)$ satisfying $F(x, t) = a$ whenever such x exists. Then, the resulting function $x = x(a, t)$ is Lipschitz (and unique) on its domain.

To observe this fact we put, in (2.7)

$$a'' = F(x'', t''), \qquad a' = F(x', t'). \tag{2.8}$$

If $\|t'' - t'\| > \varepsilon\|x'' - x'\|$, the inequality

$$\varepsilon\|x'' - x'\| \leqslant \|t'' - t'\| + \|a'' - a'\|$$

holds trivially. In the other case, the same follows from (2.7).

Step 3. Inequality (2.7) implies the existence of some positive δ such that equation

$$F(x, \bar{t}) = a, \qquad x \in B(\bar{x}, \varepsilon)$$

has a solution whenever $\|a - \bar{a}\| \leqslant \delta$.

Let us put $t'' = t' = \bar{t}$ in (2.7). The resulting inequality then says that $F(., \bar{t})$ establishes a homeomorphism between $B(\bar{x}, \varepsilon)$ and $S := F(B(\bar{x}, \varepsilon), \bar{t})$. By the invariance of domain theorem we may therefore conclude $\bar{a} = F(\bar{x}, \bar{t}) \in \operatorname{int} S$ which makes the existence of the above δ trivial.

Note that the possible application of the invariance of domain theorem, in this context, was already seen in [3, Remark 2]. For the present proof, this application is the key.

Step 4. Suppose (2.7), and let δ be taken as in Step 3. Then, for each (a, t) satisfying $L\|t - \bar{t}\| + \|a - \bar{a}\| \leqslant \delta$ and $t \in B(\bar{t}, \varepsilon)$, where L is some Lipschitz rank of F on $B(\bar{x}, \varepsilon) \times B(\bar{t}, \varepsilon)$, the equation

$$F(x, t) = a, \qquad x \in B(\bar{x}, \varepsilon) \tag{2.9}$$

has a solution.

Recalling step 2 the solutions we are speaking about, when they will exist at all, are unique and Lipschitz. The function $\Phi = F(., \bar{t})^{-1}$ is thus continuous on $B(\bar{a}, \delta)$, and by $H_a(x) = \Phi(F(x, \bar{t}) - F(x, t) + a)$ a continuous map of $B(\bar{x}, \varepsilon)$ into itself is defined. In accordance with Brouwer's fixed-point theorem, H_a possesses a fixed point which, obviously, solves (2.9).

Summarizing we have shown that (2.3) ensures regularity.

Second part. Regularity implies (2.3). Assume the contrary, i.e., (2.7) is false for each $\varepsilon > 0$. As in Step 1 we consider sequences $\varepsilon \searrow 0$, x'', x', t'', t' and write a'', a' for the corresponding values of F. Because of

$$\begin{aligned}\|a'' - a'\| + \|t'' - t'\| &< \varepsilon(\|x'' - x'\| + \|t'' - t'\|) + \varepsilon\|x'' - x'\| \\ &< 2\varepsilon\|x'' - x'\| + \varepsilon^2\|x'' - x'\|\end{aligned}$$

the implicit function G cannot be Lipschitz near $(\bar{a}, \bar{t})$.

Third part. Equivalence of (2.4) and (2.5). Since we are dealing now with the regular case the equations (2.8) and

$$x'' = G(a'', t''), \qquad x' = G(a', t') \tag{2.10}$$

mean the same for points near $(\bar{x}, \bar{t}, \bar{a})$. When the symmetry between (2.8) and (2.10) as well as (2.4) and (2.5) is taken into consideration it will be enough to show that (2.5) implies (2.4). Hence assume (2.5). Then, there are sequences $x' \to x$, $t' \to t$, and $\lambda \searrow 0$ such that, after we put $x'' = x' + \lambda u$ and $t'' = t' + \lambda\tau$, the points a'' and a' defined by (2.8) satisfy

$$\beta := (a'' - a')/\lambda \to \alpha.$$

Because of (2.10) we may write

$$\lambda u = x'' - x' = G(a'', t'') - G(a', t') = G(a' + \lambda\alpha, t' + \lambda\tau) - G(a', t') + r, \tag{2.11}$$

where r may be estimated by using some Lipschitz rank L_G of G

$$\|r\| \leqslant L_G\|a'' - a' - \lambda\alpha\| \leqslant L_G\lambda\|\beta - \alpha\|.$$

Since $\|\beta - \alpha\| \to 0$, Eq. (2.11) yields the limit-characterization of u which we need to see that (2.4) is true. This completes the proof.

Critical Points and the Marginal Function in Parametric $C^{1,1}$-Optimization

Consider the optimization problem

$$\text{minimize } f(x, t) - \langle a, x \rangle \text{ s.t. } g(x, t) + b \leqslant 0,\ h(x, t) + c = 0, \tag{3.1}$$

where the vectors $r=(a, b, c)$ and t are parameters. Concerning the function $P=(f, g, h)$ we require that $DP(.,.)$ and $D_t DP(.,.)$ exist and are locally Lipschitz. Further, assume $g=(g_1, ..., g_{m_1})$, $h=(h_1, ..., h_{m_2})$, x, $a \in R^n$, y, $b \in R^{m_1}$, z, $c \in R^{m_2}$, and $t \in R^{m_3}$.

Let us associate, to (3.1), the matrix

$$M(x, t)=\begin{bmatrix} D_x f & D_x g & 0 & D_x h \\ -g^T & 0 & E & 0 \\ -h^T & 0 & 0 & 0 \end{bmatrix}$$

of size $(n+m_1+m_2, 1+2m_1+m_2)$, and define a vector $V(y, z)=(1, y^+, y^-, z)$ of length $1+2m_1+m_2$ by setting $y_i^+=\max\{y_i, 0\}$, $y_i^-=\min\{y_i, 0\}$. We put $s=(x, y, z)$ and form the function

$$F(s, t)=M(x, t)\, V(y, z).$$

A point s satisfying

$$F(s, t)=r \tag{3.2}$$

is called *critical* for problem (3.1).

If s is critical then (x, y^+, z) is a Karush–Kuhn–Tucker point (KKTP) for this problem; if (x, y, z) is a KKTP then $(x, y+g(x), z)$ is critical. Note that both transformations are Lipschitz.

In what follows we shall apply Theorem 1 to Eq. (3.2) at some critical point $\bar{s}$ belonging to the parameters $(0, \bar{t})$. Denote by L the Lagrangian

$$L(s, t)=f(x, t)+\sum y_i^+ g_i(x, t)+\sum z_j h_j(x, t).$$

Our main tool becomes the directional derivative

$$\begin{aligned} H(u) &:= \Delta(D_x L(., \bar{y}, \bar{z}, \bar{t}))(\bar{x}; u) \\ &= \Delta_x(D_x L)((\bar{s}, \bar{t}); u) \end{aligned}$$

which coincides with $D_x^2 L(\bar{s}, \bar{t})\, u$ in the case $P \in C^2$. We will see that the (generalized complementarity) system

$$\begin{aligned} &\alpha \in H(u)+\sum p_i D_x g_i(\bar{x}, \bar{t})+\sum w_j D_x h_j(\bar{x}, \bar{t}) \\ &\beta_i=\langle -D_x g_i(\bar{x}, \bar{t}), u\rangle+q_i \\ &\gamma_j=\langle -D_x h_j(\bar{x}, \bar{t}), u\rangle \\ &v=p+q, p_i q_i \geqslant 0, p_i \bar{y}_i^-=0, q_i \bar{y}_i^+=0 \end{aligned} \tag{3.3}$$

includes all needed informations to characterize regularity as well as the directional derivatives ΔG of the implicit function. Moreover, when F is regular at $(\bar{s}, \bar{t}, 0)$ then the marginal function

$$\varphi(r, t) = f(x, t) - \langle a, x \rangle \qquad \text{with} \quad s := (x, y, z) = G(r, t) \tag{3.4}$$

will be well defined around $(0, \bar{t})$, and system (3.3) will indicate that

$$D\varphi(r, t) = (-x, y^+, z, D_t L(s, t)). \tag{3.5}$$

Again by (3.3), we then determine $\Delta D\varphi$ and the set of scalar products $\langle d, \Delta(D\varphi)((0, \bar{t}); d)\rangle$ where $d = (\alpha, \beta, \gamma, \tau)$ is some direction of parameters. The latter set is of interest since, for each functional $J \in C^{1,1}(R^n, R)$ and any points $x^1, x^2 \in R^n$, the following Taylor expansion holds.

There is some $\theta \in (0, 1)$ such that (see [16, Theorem 3])

$$\begin{aligned} &J(x^2) - J(x^1) - DJ(x^1)(x^2 - x^1) \\ &\qquad \in (1/2)\langle x^2 - x^1, \Delta(DJ)(x^1 + \theta(x^2 - x^1); x^2 - x^1)\rangle. \end{aligned}$$

Finally, it will turn out that the regularity condition (2.3), in the present case, can be reformulated by means of the tangent space

$$T = \{u / \langle D_x \bar{h}_j, u\rangle = 0, \langle D_x \bar{g}_i, u\rangle = 0 \quad \text{if} \quad \bar{y}_i > 0\}$$

with derivatives at $(\bar{x}, \bar{t})$, and the normal cones

$$K(u) = \left\{ \xi / \xi = \sum_{\bar{y}_i \geqslant 0} \lambda_i D_x \bar{g}_i + \sum \mu_j D_x \bar{h}_j \right.$$
$$\left. \text{where} \quad \lambda_i \langle D_x \bar{g}_i, u\rangle \leqslant 0 \quad \text{if} \quad \bar{y}_i = 0 \right\}.$$

Let us start to deal with system (3.3).

By Π we denote its set of solutions $\pi = (\alpha, \beta, \gamma, u, v, w, p, q)$ and define the projections

$$\begin{aligned} &R(\alpha, \beta, \gamma) = \{(u, v, w, p, q) / \pi \in \Pi\} \\ &Q(u, v, w) = \{(\alpha, \beta, \gamma) / \text{there are } p, q \text{ such that } \pi \in \Pi\}. \end{aligned}$$

The meaning of these setting becomes clear after studying the directional derivatives

$$\Delta F := \Delta F((\bar{s}, \bar{t}); (\sigma, \tau)); \qquad \sigma := (u, v, w)$$

according to the chain-rules derived in [16]. There is verified that, because of the special structure of F, equation

$$\Delta F = \Delta_s F((\bar{s}, \bar{t}); \sigma) + D_t F(\bar{s}, \bar{t})\, \tau$$

holds true. A further chain-rule may be used to obtain

$$\Delta_s F((\bar{s}, \bar{t}); \sigma) = \Delta_x M((\bar{x}, \bar{t}); u)\, V(\bar{y}, \bar{z}) + M(\bar{x}, \bar{t})\, \Delta V((\bar{y}, \bar{z}); (v, w)). \quad (3.6)$$

Within V, the only part of interest is the function $C(y) = (y^+, y^-)$ whose derivatives $\Delta C(\bar{y}; v)$ coincide with $\partial C(\bar{y})\, v$ and are given by the fourth row in (3.3).

After studying the right-hand side of (3.6) we thus observe

$$\Delta_s F((\bar{s}, \bar{t}); \sigma) = Q(u, v, w).$$

The regularity condition (2.3) means nothing but

$$0 \notin Q(\sigma)\ \forall \sigma \neq 0 \text{ or, equivalently, } R(0) = \{0\}.$$

In [16, Theorem 4] we have shown that this condition can be split into the two requirements

(LICQ) The gradients $D_x h_j(\bar{x}, \bar{t})$ and $D_x g_i(\bar{x}, \bar{t})$ $(\bar{y}_i \geq 0)$ are linearly independent

(SOC) $K(u) \cap H(u) = \varnothing$ for all $u \in T \setminus \{0\}$.

In view of Theorem 1 (iii) we thus derived

THEOREM 2. *The function F is regular at $(\bar{s}, \bar{t}, 0)$ if and only if the conditions* (LICQ) *and* (SOC) *are satisfied. In this case, the directional derivatives of the implicit function (critical point map) $G = G(r, t)$ are given by $\Delta G((0, \bar{t}); (\alpha, \beta, \gamma, \tau)) = Q^{-1}((\alpha, \beta, \gamma) - D_t F(\bar{x}, \bar{t})\, \tau)$.*

Without going into the details we have to note that, for proving (3.6) in [16], something more has been verified: Given any solution of system (3.3) there are sequences $\lambda \searrow 0$ and $(x, y, z) \to (\bar{x}, \bar{y}, \bar{z})$ such that both

$$(\alpha, \beta, \gamma) = \lim (1/\lambda)[F(x + \lambda u, y + \lambda v, z + \lambda w, \bar{t}) - F(x, y, z, \bar{t})]$$

and

$$(p, q) = \lim (1/\lambda)[C(y + \lambda v) - C(y)].$$

In the regular case, this yields the existence of sequences $\lambda \searrow 0$ and $(a, b, c) \to 0$ such that

$$(u, v, w) = \lim (1/\lambda)[G(a + \lambda\alpha, b + \lambda\beta, c + \lambda\gamma, \bar{t}) - G(a, b, c, \bar{t})]$$

and

$$(p, q) = \lim(1/\lambda)[Y(a+\lambda\alpha, b+\lambda\beta, c+\lambda\gamma, \bar{t}) - Y(a, b, c, \bar{t})],$$

where Y denotes the corresponding split Lagrange multiplier (y^+, y^-). In order to prove this statement it suffices to put $(a, b, c) = F(x, y, z, \bar{t})$ and to determine the limits in question. As a consequence, we are able to describe the directional derivatives of the function (G, Y) that associates, to (r, t) near $(0, \bar{t})$, the vector (x, y, z, y^+, y^-), $s = G(r, t)$.

$$\Delta(G, Y)((0, \bar{t}); (\alpha, \beta, \gamma, \tau)) = R((\alpha, \beta, \gamma) - D_t F(\bar{s}, \bar{t})\,\tau). \tag{3.7}$$

At this point, we are prepeared to deal with the marginal function (3.4) under regularity assumption. Let $d = (\alpha, \beta, \gamma, \tau)$. Since Df is continuous the set $\Delta\varphi((0, \bar{t}); d)$ consists of all vectors $\psi = \langle D_x f(\bar{x}, \bar{t}), u\rangle + D_t f(\bar{x}, \bar{t})\,\tau - \langle \alpha, \bar{x}\rangle$ with $(u, v, w) \in \Delta G((0, \bar{t}); d)$. After we substitute

$$D_x \bar{f} = -\sum \bar{y}_i^+ D_x \bar{g}_i - \sum \bar{z}_j D_x \bar{h}_j$$

and study (3.3) with

$$(\hat{\alpha}, \hat{\beta}, \hat{\gamma}) = (\alpha, \beta, \gamma) - D_t F(\bar{s}, \bar{t})\,\tau,$$

it is seen that ψ does not depend on the concrete choice of (u, v, w): $\psi = -\langle \alpha, \bar{x}\rangle + \langle \beta, \bar{y}^+\rangle + \langle \gamma, \bar{z}\rangle + D_t L(\bar{s}, \bar{t})\,\tau$. Since regularity, by definition, will also hold for the points (s, t, r) being under consideration and since the linear parameter r may be seen as included in the function $P = (f, g, h)$ we thus derived (3.5).

The directional derivatives of the function

$$(r, t) \mapsto (-x, y^+, z, D_t L(s, t)), \qquad s = G(r, t)$$

are now given via (3.7). We obtain

$$\begin{aligned} \Delta(D\varphi)((0, \bar{t}); d) = \{ &(-u, p, w, (D_x, D_{y^+}, D_z, D_t)\, D_t\, L(\bar{s}, \bar{t})(u, p, w, \tau))/ \\ &(u, v, w, p, q) \in R(\hat{\alpha}, \hat{\beta}, \hat{\gamma})\}. \end{aligned} \tag{3.8}$$

This allows us, after some simple calculation, to characterize the associated set of scalar products

$$\begin{aligned} &\langle d, \Delta(D\varphi)((0, \bar{t}); d)\rangle \\ &\quad = \{\langle -u, \hat{\alpha}\rangle + \langle p, \hat{\beta}\rangle + \langle w, \hat{\gamma}\rangle + \langle \tau, D_t^2 L(\bar{s}, \bar{t})\,\tau\rangle/ \\ &\qquad (u, v, w, p, q) \in R(\hat{\alpha}, \hat{\beta}, \hat{\gamma})\}. \end{aligned} \tag{3.9}$$

Let us, finally, deal with the strict complementarity condition $\bar{y}_i \neq 0$ (for all

i). To do so we omit redundant constraints by supposing $\bar{y} \geqslant 0$, assume regularity of F at $(\bar{s}, \bar{t}, 0)$, and fix the nonlinear parameter $t \equiv \bar{t}$. The map $D\varphi$, considered as a function from $R^{n+m_1+m_2}$ into itself, is now given by

$$D\varphi(a, b, c) = (-x, y^+, z), \qquad s = G(r, \bar{t}).$$

Of course, $D\varphi$ cannot be regular at $(0, 0, 0, -\bar{x}, \bar{y}^+, \bar{z})$ if some $\bar{y}_i$ equals zero. On the other hand, if $\bar{y}_i > 0$ for all i, then formula (3.8) and system (3.3) show that

$$0 \in \Delta(D\varphi)(0; (\alpha, \beta, \gamma)) \quad \text{means} \quad 0 \in R(\alpha, \beta, \gamma)$$

and implies $(\alpha, \beta, \gamma) = 0$. Therefore, the function $D\varphi$ is then regular at the point of interest.

Concluding Remarks

1. Since $\langle \xi, u \rangle \leqslant 0$ is true for any $u \in T$ and $\xi \in K(u)$, the condition (SOC) of Theorem 2 will hold whenever the following strong second-order condition is satisfied

$$\langle \xi, u \rangle > 0 \qquad \text{if} \quad u \in T \setminus \{0\} \text{ and } \xi \in H(u).$$

We refer to [14] for consequences of this condition concerning strict local minimizers.

2. Let x be some regular zero of $f \in C^{0,1}(R, R)$. In contrast with the C^1-case, we then may not conclude that Newton's method (to determine x) will locally converge even if f is continuously differentiable at each point generated by the procedure. An example where this method fails to converge with almost all initial points, the reader finds in [17, Sect. 2.3].

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